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IMPULSIVE LOADING OF IDEAL FIBRE-REINFORCED RIGID-PLASTIC BEAMS—II

BEAM WITH END SUPPORTS

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Abstract—The theory outlined in Part I is applied to the problem of a beam, supported at both ends, struck transversely at any point by a mass which subsequently adheres to the beam. For sufficiently long beams, the resulting moving discontinuities in slope and velocity do not propagate to the ends of the beam, and for this case the solution is obtained for the non-linear strain-hardening law used in Part I. A simpler approximate solution is derived for the case of low impact velocity and/or slight strain hardening. For shorter beams, the propagating discontinuities may undergo one or more successive reflections at the end points and at the point of initial impact. The deformation after such reflections is analysed, for linear strain-hardening, in the final two sections. Some particularly simple results are found in the case of central impact.

1. INTRODUCTION

The general theory of the behaviour of ideal fibre-reinforced rigid-plastic beams was outlined in Part I[1], which also gives references to other work. In this paper we apply this theory to a beam, supported at both ends, struck by a mass $2M$ at any point. The support conditions (for example, simple support or clamping) at the ends do not affect the main features of the solution, because the fibre tensions, which do not influence the deformation, adjust to equilibriate any couples which are applied to the ends of the beam.

The notation is as in Part I. Initially the beam lies along the X-axis from $X = -L$ to $X = L$ and it is struck at time $T = 0$ at the point $X = X_0 = x_0 L$ by a mass $2M$ moving in the Y-direction; without loss of generality it is assumed that $x_0 \geq 0$. The mass subsequently adheres to the beam. We seek solutions in which initially a segment of the beam $X_0 - A(T) < X < X_0 + A(T)$ moves as a rigid body with speed $V(T)$ in the Y-direction, with discontinuities in slope and speed at $X_0 \pm A(T)$ propagating to the right and left respectively. The remainder of the beam is at rest. The configuration is illustrated in Fig. 1.

For sufficiently small values of α, $\omega\beta$ and X_0, the beam comes to rest before the discontinuity at $X = A(T)$ reaches the end $X = L$ of the beam. The solution for this case (a long beam) is given in Sections 2 and 3, for general values of n. For larger values of α, $\omega\beta$ and X_0 (a short beam), the right-hand discontinuity is reflected at $X = L$. For sufficiently large values of the parameters, further reflections may occur at $X = -L$, $X = X_0$ and $X = L$. The theory for short beams is described in Sections 4 and 5, for the case $n = 1$ of a linear strainhardening material.

In [2], Jones has given the solution for a long beam in the linear strain-hardening case, and has compared the results with the corresponding results for an isotropic beam.

A discussion of the solutions is given at the end of Part III.

2. GENERAL SOLUTION FOR A LONG BEAM

The assumed configuration is illustrated in Fig. 1. The slope of the right-hand half of the moving segment $A'A$ is denoted by $f_1(x)$ so that, in the non-dimensional variables defined in Part I,

$$\gamma = \begin{cases} f_1(x), & x_0 < x < x_0 + a(t), \\ -f_1(2x_0 - x), & x_0 - a(t) < x < x_0, \\ 0, & -1 < x < x_0 - a(t) \text{ and } x_0 + a(t) < x < 1. \end{cases} \tag{2.1}$$

Here $x_0 = X_0/L$ and, as in Part I, it is assumed that Q_p has the form

$$Q_p = Q_0 + Q_1|\gamma|^n.$$

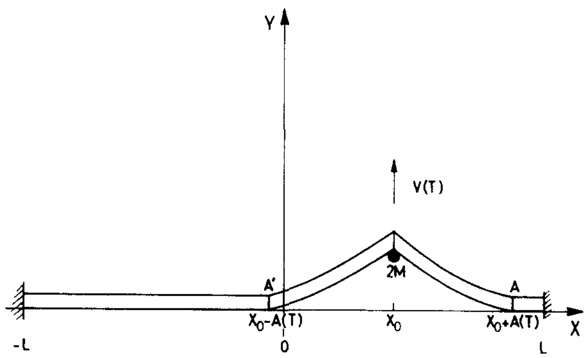

Fig. 1. Impact of a long beam with fixed ends. Assumed form of deformation.

Then the governing equations are as follows:

(a) Equation of motion of the segment $x_0 - a < x < x_0 + a$:

$$\beta^2(\alpha + a)\dot{v} = -1 - \omega^{2n}\{-f_1(x_0 + a)\}^n. \tag{2.2}$$

(b) Dynamic jump condition at $x = x_0 + a$:

$$\beta^2 \dot{a} v = \omega^{2n}\{-f_1(x_0 + a)\}^n. \tag{2.3}$$

(c) Kinematic jump condition at $x = x_0 + a$:

$$v = -\dot{a} f_1(x_0 + a). \tag{2.4}$$

Relations similar to (2.3) and (2.4) are given by the jump conditions at $x = x_0 - a$, but these merely confirm the symmetry of the deformed segment about $x = x_0$. Note that (2.2), (2.3) and (2.4) can be obtained from (3.8), (3.10) and (3.11) of Part I by setting $v_1 = v$, $v_2 = 0$ and $f(x) = f_1(x)$. The initial conditions are

$$v = 1, \qquad a = 0 \qquad \text{when} \qquad t = 0.$$

By adding (2.2) and (2.3), then integrating and inserting the initial conditions, we obtain

$$(a + \alpha)v = \alpha - \beta^{-2}t. \tag{2.5}$$

This expresses balance of linear momentum for the whole beam. From (2.3) and (2.4)

$$\beta^2 \dot{a}^{n+1} = \omega^{2n} v^{n-1}. \tag{2.6}$$

Then from (2.5) and (2.6)

$$(a + \alpha)^{q-1}\dot{a} = \beta^{q-2}\omega^q(\alpha - \beta^{-2}t)^{q-1}, \tag{2.7}$$

where, as in Part I, $q = 2n/(n + 1)$. With the initial conditions, (2.7) integrates to give

$$(a + \alpha)^q - \alpha^q = (\omega\beta)^q\{\alpha^q - (\alpha - \beta^{-2}t)^q\}. \tag{2.8}$$

We denote

$$t_1 = \alpha\beta^2, \qquad a_1 = \alpha[\{1 + (\omega\beta)^q\}^{1/q} - 1], \tag{2.9}$$

and then (2.8) can be written

$$\left(1+\frac{a_1}{\alpha}\right)^q-\left(1+\frac{a}{\alpha}\right)^q=(\omega\beta)^q\left(1-\frac{t}{t_1}\right)^q, \tag{2.10}$$

and we note that $a=a_1$ when $t=t_1$. Then (2.5) can be expressed as

$$\left(1+\frac{a}{\alpha}\right)v=1-\frac{t}{t_1}, \tag{2.11}$$

so that $v=0$ when $t=t_1$. From (2.10) and (2.11) we obtain v as a function of a or of t, as

$$(\omega\beta v)^q=\left\{\left(1+\frac{a_1}{\alpha}\right)^q\left(1+\frac{a}{\alpha}\right)^{-q}-1\right\} \tag{2.12}$$

$$=\frac{\left(1-\frac{t}{t_1}\right)^q}{\left\{(\omega\beta)^{-q}\left(1+\frac{a_1}{\alpha}\right)^q-\left(1-\frac{t}{t_1}\right)^q\right\}}. \tag{2.13}$$

Also from (2.3), (2.4) and (2.12)

$$\omega^2 f_1(x_0+a)=-\left\{\left(1+\frac{a_1}{\alpha}\right)^q\left(1+\frac{a}{\alpha}\right)^{-q}-1\right\}^{(2-q)/q}. \tag{2.14}$$

The deflection $Lu(x,t)$ of the beam is given by

$$u(x,t)=\begin{cases}-\displaystyle\int_x^{x_0+a(t)} f_1(\xi)\,d\xi, & x_0\leq x\leq x_0+a(t),\\ 0, & -1\leq x\leq x_0-a(t) \text{ and } x_0+a(t)\leq x\leq 1,\end{cases} \tag{2.15}$$

and

$$u(x_0-\xi,t)=u(x_0+\xi,t),\qquad 0\leq\xi\leq a(t).$$

The value of Q in the rigid segments and the tensions in the singular fibres are given by (2.12) and (2.13) of Part I. It can be verified that the yield condition is not violated in the rigid segments and that the rate of plastic working at the discontinuities is positive. These conditions are also satisfied by the solutions given later in this paper and in Part III, and they will not be explicitly mentioned in connection with each separate solution.

The first stage of the deformation terminates when either (a) the beam comes to rest. Then $v=0$, and $a=a_1$ and $t=t_1$; or (b) the discontinuity $x=x_0+a(t)$ reaches the end $x=1$ of the beam. In this case $a=1-x_0$ and, from (2.10),

$$t=t_2=t_1\left[1-(\omega\beta)^{-1}\left\{\left(1+\frac{a_1}{\alpha}\right)^q-\left(1+\frac{1-x_0}{\alpha}\right)^q\right\}^{1/q}\right]. \tag{2.16}$$

If $t_1<t_2$, or equivalently $x_0+a_1<1$, the motion is completed before the discontinuity reaches the end of the beam, and the results of this section give the complete solution. In our terminology, this is the case of a long beam. In terms of the parameters α, β, ω and x_0, the condition $t_1<t_2$ for the beam to be a long beam is, from (2.9) and (2.16),

$$(\omega\beta)^q<\left(1+\frac{1-x_0}{\alpha}\right)^q-1. \tag{2.17}$$

If $t_1 > t_2$, or

$$(\omega\beta)^q > \left(1 + \frac{1 - x_0}{\alpha}\right)^q - 1, \tag{2.18}$$

the results of this section hold for $t \leq t_2$, but for $t > t_2$ the solution takes a different form, which is considered in Sections 4 and 5.

The results simplify considerably in the case of linear strain-hardening, $q = 1$. For a long beam, and $x_0 = 0$, the solution for $q = 1$ was given by Jones[2]. For use later, we give the main results for any value of x_0 and $q = 1$. Equations (2.9) reduce to $t_1 = \alpha\beta^2$, $a_1 = \alpha\omega\beta$, so that (2.10) becomes

$$a = \omega t/\beta. \tag{2.19}$$

Equations (2.13) and (2.14) reduce to

$$\omega\beta v = \frac{\alpha\omega\beta - (\omega t/\beta)}{\alpha + (\omega t/\beta)}, \tag{2.20}$$

$$\omega^2 f_1(x_0 + a) = -\left\{\frac{\alpha(1 + \omega\beta)}{a + \alpha} - 1\right\} = -\frac{(\alpha\omega\beta - a)}{a + \alpha}. \tag{2.21}$$

Hence from (2.15) and (2.19) the deflection is given by

$$\omega^2 u(x, t) = \left\{\alpha(1 + \omega\beta) \log\left(\frac{\omega\beta^{-1}t + \alpha}{|x - x_0| + \alpha}\right) + |x - x_0| - \omega\beta^{-1}t\right\}, \tag{2.22}$$

for $|x - x_0| \leq \omega\beta^{-1}t$, and $u(x, t) = 0$ elsewhere. The condition (2.17) for a long beam reduces to $\alpha\omega\beta < 1 - x_0$; if this is satisfied (2.19)–(2.22) give the solution up to the time $t_1 = \alpha\beta^2$ at which the beam comes to rest. If $\alpha\omega\beta > 1 - x_0$, the above solution is valid for $t \leq t_2 = \omega^{-1}\beta(1 - x_0)$, after which the solution takes the form described in Sections 4 and 5.

The solution of this section also gives the solution for a long cantilever beam struck at its tip. If we set $x_0 = 0$, so that the solution is symmetrical about $x = 0$, then the solution in $0 < x < 1$ can be interpreted as that for a cantilever beam of length L, built in at $X = L$ and struck at its tip $X = 0$ by a mass M moving in the Y-direction with speed V_0. The solution is complete provided that the condition (2.17) (with $x_0 = 0$) is satisfied, so that the discontinuity does not propagate to $x = 1$. This problem also was discussed by Jones[2] for the case of linear strain-hardening. The case in which (2.17) is not satisfied is considered in Section 4. More general problems for cantilever beams are described in Part III.

3. SOLUTION FOR $\omega\beta \ll 1$

As was mentioned in Section 7 of Part I, the condition $\omega\beta \ll 1$ requires low impact velocity or slight strain-hardening or both. It is also assumed that q^{-1} is of order one. Then (2.9) gives approximately

$$a_1 = \alpha q^{-1}(\omega\beta)^q, \tag{3.1}$$

so that $a_1/\alpha \ll 1$, and hence $a/\alpha \ll 1$. Therefore, to leading order in $\omega\beta$, (2.10) and (2.12)–(2.14) become

$$a_1 - a = \frac{\alpha}{q}(\omega\beta)^q\left(1 - \frac{t}{t_1}\right)^q, \tag{3.2}$$

$$(\omega\beta v)^q = q\alpha^{-1}(a_1 - a), \tag{3.3}$$

$$v = 1 - \frac{t}{t_1}, \tag{3.4}$$

$$\omega^2 f_1(x_0 + a) = -\left\{\frac{q(a_1 - a)}{\alpha}\right\}^{(2-q)/q}. \tag{3.5}$$

Hence, from (2.15), for $|x - x_0| < a(t)$

$$\omega^2 u(x, t) = \frac{\alpha}{2}\left(\frac{q}{\alpha}\right)^{2/q} [\{a_1 - |x - x_0|\}^{2/q} - \{a_1 - a(t)\}^{2/q}]. \tag{3.6}$$

The condition (2.17) shows that, when $\omega\beta \ll 1$, the beam may be regarded as a long beam except possibly when $\alpha \gg 1$ (a heavy striker) or $1 - x_0 \ll 1$, in which case the point of impact is close to the end of the beam. Neither of these cases seems to be of major interest, and they are excluded from further consideration.

It is interesting to note that the maximum final deflection, which is obtained by setting $x = x_0$ and $a(t) = a_1$ in (3.6), is given by $u = \frac{1}{2}\alpha\beta^2$, and is independent of the value of the work-hardening parameter q.

4. SHORT BEAM UNDER CENTRAL IMPACT

For central impact $x_0 = 0$. Then the condition (2.18) for a short beam reduces to

$$(\omega\beta)^q > (1 + \alpha^{-1})^q - 1. \tag{4.1}$$

For $t < t_2$ the solution is given by (2.12), (2.13) and (2.14) (with $x_0 = 0$). At time $t = t_2$ the two propagating discontinuities reach the two ends of the beam simultaneously. At this instant the slope of the beam is given by (2.14) as

$$\omega^2 f_1(x) = -\left\{\left(1 + \frac{a_1}{\alpha}\right)^q \left(1 + \frac{x}{\alpha}\right)^{-q} - 1\right\}^{(2-q)/q}, \qquad (0 < x \leq 1) \tag{4.2}$$

with $f_1(-x) = -f_1(x)$. By symmetry it is sufficient to consider the range $0 < x \leq 1$.

For $t > t_2$ we seek solutions in which a central segment $-a(t) < x < a(t)$ of the beam moves as a rigid body, but now the discontinuities at $x = \pm a(t)$ have been reflected at $x = \pm 1$ and move inwards towards the centre of the beam. The outer segments are at rest, and a deformation occurs as a discontinuity crosses a section of the beam. The assumed configuration of the beam is shown schematically in Fig. 2, and the propagation of the discontinuities in the (x, t) plane is illustrated in Fig. 3. The slope of the beam in the stationary segment $a(t) < x < 1$ is denoted by $\gamma = g_1(x)$.

For $t > t_2$, the governing equations are:

(a) Equation of motion of the segment $-a < x < a$:

$$\beta^2(\alpha + a)\dot{v} = -1 - \omega^{2n}\{-f_1(a)\}^n. \tag{4.3}$$

(b) Dynamic jump condition at $x = a$:

$$\beta^2 \dot{a} v = \omega^{2n}[\{-f_1(a)\}^n - \{-g_1(a)\}^n]. \tag{4.4}$$

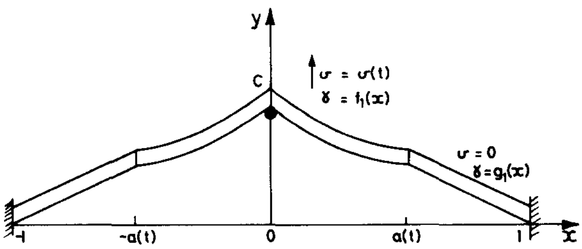

Fig. 2. Central impact of a short beam with fixed ends. Assumed form of deformation after reflection of the discontinuities.

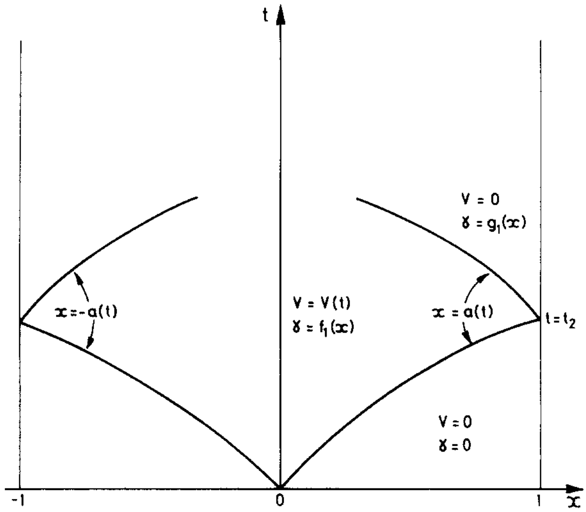

Fig. 3. Central impact of a short beam with fixed ends. Trajectories of the discontinuities in the (x, t) plane.

(c) Kinematic jump condition at $x = a$:

$$v = -\dot{a}\{f_1(a) - g_1(a)\}. \tag{4.5}$$

In these, $f_1(x)$ is given by (4.2) and it is necessary to determine $a(t)$, $v(t)$ and $g(x)$ subject to the conditions $a(t_2) = 1$ and, from (2.12),

$$\{\omega\beta v(t_2)\}^q = \left(\frac{\alpha + a_1}{\alpha + 1}\right)^q - 1. \tag{4.6}$$

An analytical solution of these equations does not seem to be possible for general values of n. However, they simplify a great deal in the case of linear strain-hardening, $n = 1$, $q = 1$, and for the remainder of this paper we consider this case only. The essential simplification is that for linear strain-hardening $\dot{a}$ is constant in magnitude, with value $\dot{a}^2 = \omega^2\beta^{-2}$. The trajectories of the discontinuities in the (x, t) plane then become straight lines as illustrated in Fig. 4, which also shows the trajectories if the discontinuities are further reflected at $x = 0$ and $x = \pm 1$. These multiple reflections continue until $v = 0$ and the entire beam comes to rest.

For $n = 1$, $t_2 = \beta/\omega$ and $a_1 = \alpha\omega\beta$. Then eqn (4.2) gives

$$\omega^2 f_1(x) = -\left\{\frac{\alpha(1 + \omega\beta)}{x + \alpha} - 1\right\} = -\frac{\alpha\omega\beta - x}{x + \alpha}, \tag{4.7}$$

and (4.6) reduces to

$$\omega\beta v(t_2) = \frac{\alpha\omega\beta - 1}{\alpha + 1}. \tag{4.8}$$

In the range $\beta/\omega < t < 2\beta/\omega$, we have

$$a = 2 - \omega t/\beta, \qquad \dot{a} = -\omega/\beta. \tag{4.9}$$

Hence (4.3) becomes

$$\beta^2(2 + \alpha - \omega t/\beta)\dot{v} = -\frac{\alpha(1 + \omega\beta)}{2 + \alpha - \omega t/\beta} \tag{4.10}$$

and (4.4) and (4.5) both reduce to

$$\omega\beta v = \omega^2\{f_1(a) - g_1(a)\}. \tag{4.11}$$

By integrating (4.10) and using the initial condtion (4.8), we obtain

$$\omega\beta v = \frac{2\alpha\omega\beta + \alpha - 1}{\alpha + 1} - \frac{\alpha(1 + \omega\beta)}{2 + \alpha - (\omega t/\beta)} \tag{4.12}$$

$$= \frac{2\alpha\omega\beta + \alpha - 1}{\alpha + 1} - \frac{\alpha(1 + \omega\beta)}{a + \alpha}. \tag{4.13}$$

Hence, from (4.7), (4.11) and (4.13),

$$\omega^2 g_1(x) = -\frac{2(\alpha\omega\beta - 1)}{\alpha + 1}. \tag{4.14}$$

Thus we have the rather remarkable result that the slope is constant in the outer segments $|a| < |x| < 1$. The deflection is therefore easily calculated in these segments.

From (4.13) it follows that $v = 0$ when

$$a = \frac{\alpha\{2 - \omega\beta(\alpha - 1)\}}{2\alpha\omega\beta + \alpha - 1}. \tag{4.15}$$

The beam comes to rest, and the deformation ceases, before the discontinuity returns to the point of impact, if (4.15) yields a positive value of a. For $q = 1$, (4.1) reduces to $\alpha\omega\beta > 1$. Hence the denominator of (4.15) is positive, and the condition for a to be positive is

$$\omega\beta(\alpha - 1) < 2. \tag{4.16}$$

We note that this condition is always satisfied if $\alpha < 1$, so the discontinuity never returns to the point of impact if the mass of the striker is less than that of the beam.

If $\omega\beta(\alpha - 1) > 2$, then for $\omega t > 2\beta$ discontinuities again propagate outwards from the centre of the beam, as shown in Fig. 4. As the magnitude of the slope of the beam is uniform at $\omega t = 2\beta$, these discontinuities propagate into regions in which Q_p is constant. Hence the theory for $2\beta < \omega t < 4\beta$ is essentially the same as that for $0 < \omega t < 2\beta$, and the solution is readily obtained by making the appropriate substitutions. For sufficiently large values of α and $\omega\beta$, further reflections may occur, but these also involve no essentially new situations.

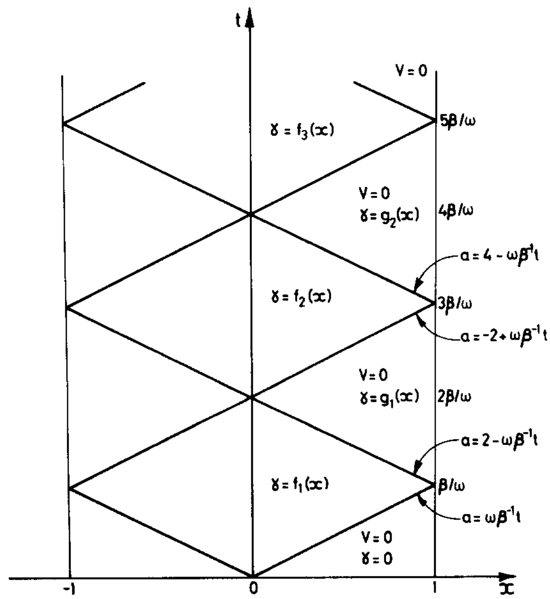

Fig. 4. Central impact of a short beam with fixed ends. Trajectories of the discontinuities in the (x, t) plane for linear strain-hardening.

The solution of this section can also be interpreted as the solution for a cantilever beam of length L, built in at $X = L$ and struck at its tip $X = 0$ by a mass M travelling with speed V_0 in the Y-direction.

5. SHORT BEAM WITH IMPACT AT $X = X_0$

We now return to the general case with impact at $X = X_0 = x_0 L$ and the condition (2.18) satisfied. As in the case considered in Section 4, it is straightforward to formulate the equations for $t > t_2$ and general values of n, but analytical solution seems to be feasible only for the case $n = 1$ of linear strain-hardening. In this case the trajectories of the discontinuities in the (x, t) plane are straight lines with slope $\pm\beta/\omega$. The discontinuities may not cross the section $x = x_0$, for doing so would imply an instantaneous change in the speed of the mass $2M$, and this can only be effected by the application of an impulsive force. Therefore we seek solutions in which the discontinuities are successively reflected each time they meet either the ends $x = \pm 1$ of the beam, or the mass at $x = x_0$, until the beam eventually comes to rest.

The motion is illustrated in the (x, t) plane in Fig. 5. At time t, there is a discontinuity at $x = x_0 + a(t)$ in the range $x_0 < x \leq 1$, and a discontinuity at $x = x_0 - b(t)$ in the range $-1 \leq x < x_0$. Each of these discontinuities propagates alternately outwards and inwards across its range with constant speed ω/β. The segment $x_0 - b < x < x_0 + a$ moves as a rigid body and the segments $x_0 + a < x < 1$ and $-1 < x < x_0 - b$ are at rest. As illustrated in Fig. 5, the slope γ is denoted $f_r(x)$, $g_r(x)$, $h_s(x)$, $k_s(x)$, $(r, s = 1, 2, 3, \ldots)$, in the various regions of the (x, t) plane.

Given a time t, there are values of r and s, which can be read off from Fig. 5, such that

$$\gamma = \begin{cases} h_s(x), & x_0 - b(t) < x < x_0, \\ f_r(x), & x_0 < x < x_0 + a(t). \end{cases} \tag{5.1}$$

Then the equation of motion at time t of the segment $x_0 - b < x < x_0 + a$ is

$$\beta^2(2\alpha + a + b)\dot{v} = -2 - \omega^2\{-f_r(x_0 + a) + h_s(x_0 - b)\}. \tag{5.2}$$

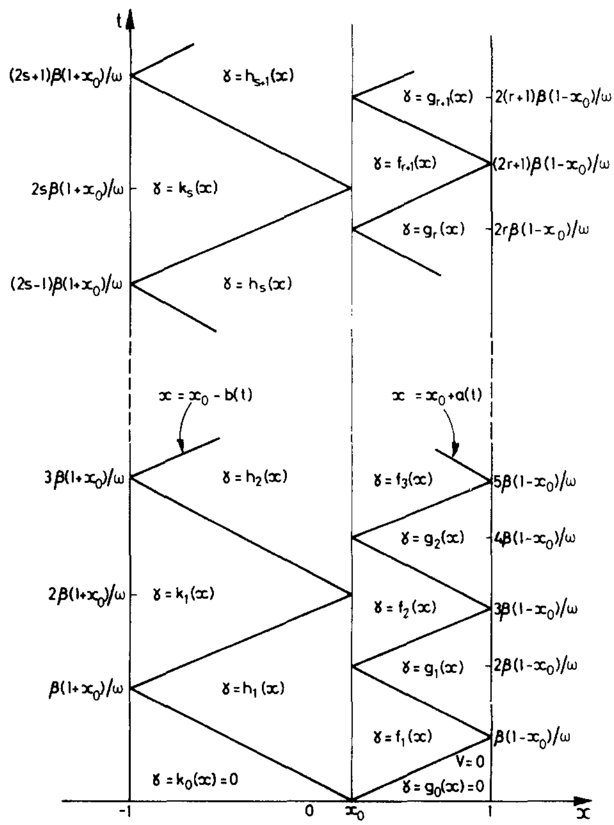

Fig. 5. Non-central impact of a short beam with fixed ends. Trajectories of the discontinuities in the (x, t) plane for linear strain-hardening.

For linear hardening, the kinematic and dynamic jump conditions at $x = x_0 + a(t)$ become identical, and are (see Fig. 5)

$$\omega\beta v = \omega^2\{-f_r(x_0+a)+g_{r-1}(x_0+a)\} \quad \text{when} \quad \dot{a} = \omega/\beta > 0, \tag{5.3}$$

$$\omega\beta v = \omega^2\{f_r(x_0+a)-g_r(x_0+a)\} \quad \text{when} \quad \dot{a} = -\omega/\beta < 0. \tag{5.4}$$

Similarly, the jump conditions at $x_0 - b(t)$ give

$$\omega\beta v = \omega^2\{h_s(x_0-b)-k_{s-1}(x_0-b)\} \quad \text{when} \quad \dot{b} = \omega/\beta > 0, \tag{5.5}$$

$$\omega\beta v = \omega^2\{-h_s(x_0-b)+k_s(x_0-b)\} \quad \text{when} \quad \dot{b} = -\omega/\beta < 0. \tag{5.6}$$

There are four cases, which correspond to the four possible combinations of signs of $\dot{a}$ and $\dot{b}$. In general, these cases do not follow each other in any regular sequence, and the sequence in which they do occur depends on the value of x_0. As an example, we obtain the solution for $0 < x_0 < \frac{1}{3}$ (which is the case illustrated in Fig. 5) up to time $t = 2\beta(1-x_0)/\omega$.

For $0 < \omega t < \beta(1-x_0)$, the solution is that given in Section 2, namely

$$\omega\beta v = \frac{\alpha\omega\beta - (\omega t/\beta)}{\alpha + (\omega t/\beta)}, \tag{5.7}$$

$$\omega^2 f_1(x_0+a) = -\frac{\alpha\omega\beta - a}{a+\alpha}, \tag{5.8}$$

$$\omega^2 h_1(x_0-b) = \frac{\alpha\omega\beta - b}{b+\alpha}. \tag{5.9}$$

Hence

$$\omega^2 f_1(x) = -\frac{\alpha\omega\beta - x + x_0}{x - x_0 + \alpha}, \qquad (x_0 < x < 1), \tag{5.10}$$

$$\omega^2 h_1(x) = \frac{\alpha\omega\beta - x_0 + x}{x_0 - x + \alpha}, \qquad (-1 + 2x_0 < x < x_0). \tag{5.11}$$

If $\alpha\omega\beta > 1 - x_0$, the motion continues for $\omega t > \beta(1-x_0)$. Since, by assumption, $0 < x_0 < \frac{1}{3}$, the discontinuity at $x = x_0 - b$ reaches the end $x = -1$ before the reflected discontinuity at $x = x_0 + a$ reaches $x = x_0$. Hence the second stage of the deformation takes place in the time interval $\beta(1-x_0) < \omega t < \beta(1+x_0)$, and in this interval, from Fig. 5

$$a = -\omega t/\beta + 2(1-x_0), \qquad b = \omega t/\beta. \tag{5.12}$$

Then (5.2), (5.4) and (5.5), with $r = 1$, $s = 1$, give

$$2\beta^2(\alpha + 1 - x_0)\dot{v} = -2 - \omega^2\{-f_1(x_0+a) + h_1(x_0-b)\}, \tag{5.13}$$

$$\omega\beta v = \omega^2\{f_1(x_0+a) - g_1(x_0+a)\}, \tag{5.14}$$

$$\omega\beta v = \omega^2 h_1(x_0-b). \tag{5.15}$$

It is convenient to use a rather than t as the independent variable. By eliminating $h_1(x_0-b)$ from (5.13) and (5.15), and introducing the expression (5.8) for $f_1(x_0+a)$, we obtain

$$\omega\beta\left\{2(\alpha+1-x_0)\frac{\mathrm{d}v}{\mathrm{d}a} - v\right\} = 1 + \frac{\alpha(1+\omega\beta)}{a+\alpha}. \tag{5.16}$$

This is a linear first order equation for v, to be solved subject to the condition that v is continuous at $t = \beta(1-x_0)/\omega$. From (5.7), this condition is

$$\omega\beta v = \frac{\alpha\omega\beta - 1 + x_0}{\alpha + 1 - x_0}, \quad \text{when} \quad a = 1 - x_0. \tag{5.17}$$

The required solution is

$$\omega\beta v = -1 + \frac{\alpha(1+\omega\beta)}{2(\alpha+1-x_0)} \exp\left\{\frac{a+\alpha}{2(\alpha+1-x_0)}\right\}\left[2\exp\left(-\tfrac{1}{2}\right) + E_1\left(\tfrac{1}{2}\right) - E_1\left\{\frac{a+\alpha}{2(\alpha+1-x_0)}\right\}\right], \tag{5.18}$$

where $E_1(\xi)$ is the exponential integral

$$E_1(\xi) = \int_\xi^\infty \frac{1}{\eta} e^{-\eta}\, d\eta, \tag{5.19}$$

which is tabulated in, for example, Abramowitz and Stegun (Ref. [7] of Part I).

From (5.10), (5.14) and (5.18) there follows

$$\omega^2 g_1(x) = \frac{2(x-x_0) + \alpha(1-\omega\beta)}{x - x_0 + \alpha} - \frac{\alpha(1+\omega\beta)}{2(\alpha+1-x_0)} \exp\left\{\frac{x-x_0+\alpha}{2(\alpha+1-x_0)}\right\}\left[2\exp\left(-\tfrac{1}{2}\right) + E_1\left(\tfrac{1}{2}\right) - E_1\left\{\frac{x-x_0+\alpha}{2(\alpha+1-x_0)}\right\}\right], \tag{5.20}$$

for $1-2x_0 < x < 1$. In the interval $\beta(1-x_0) < \omega t < \beta(1+x_0)$ we have, from (5.12) or from Fig. 5, $a = -b + 2(1-x_0)$. Hence from (5.15) and (5.18)

$$\omega^2 h_1(x) = -1 + \frac{\alpha(1+\omega\beta)}{2(\alpha+1-x_0)} \exp\left\{\frac{\alpha + 2(1-x_0) - (x_0 - x)}{2(\alpha+1-x_0)}\right\}\left[2\exp\left(-\tfrac{1}{2}\right) + E_1\left(\tfrac{1}{2}\right) - E_1\left\{\frac{\alpha + 2(1-x_0) - (x_0-x)}{2(\alpha+1-x_0)}\right\}\right] \tag{5.21}$$

for $-1 < x < -1 + 2x_0$.

The deformation terminates when $v = 0$. If $v = 0$ for a value of t less than $\beta(1+x_0)/\omega$, then the above results give the complete solution. Fig. 6 shows the variation of v with t for $x_0 = 0.2$ and a number of values of α and $\alpha\omega\beta$, and for t not exceeding $\beta(1+x_0)/\omega$. In Fig. 6, v is given by (5.7) for $0 \leq \omega t/\beta \leq 1 - x_0$, and by (5.18) for $1 - x_0 \leq \omega t/\beta \leq 1 + x_0$.

If (5.18) gives $v > 0$ at $t = \beta(1+x_0)/\omega$ (that is, $a = 1 - 3x_0$), then the deformation proceeds to the third stage which, from Fig. 5, takes place in the time interval $\beta(1+x_0) < \omega t < 2\beta(1-x_0)$. In this interval

$$a = 2(1-x_0) - \omega t/\beta, \qquad b = 2(1+x_0) - \omega t/\beta, \tag{5.22}$$

and (5.2), (5.4) and (5.6) give

$$2\beta^2(\alpha + 2 - \omega t/\beta)\dot{v} = -2 - \omega^2\{-f_1(x_0+a) + h_1(x_0-b)\}, \tag{5.23}$$

$$\omega\beta v = \omega^2\{f_1(x_0+a) - g_1(x_0+a)\}, \tag{5.24}$$

$$\omega\beta v = \omega^2\{k_1(x_0-b) - h_1(x_0-b)\}. \tag{5.25}$$

Since $f_1(x)$ and $h_1(x)$ have been determined already, eqn (5.23), together with the

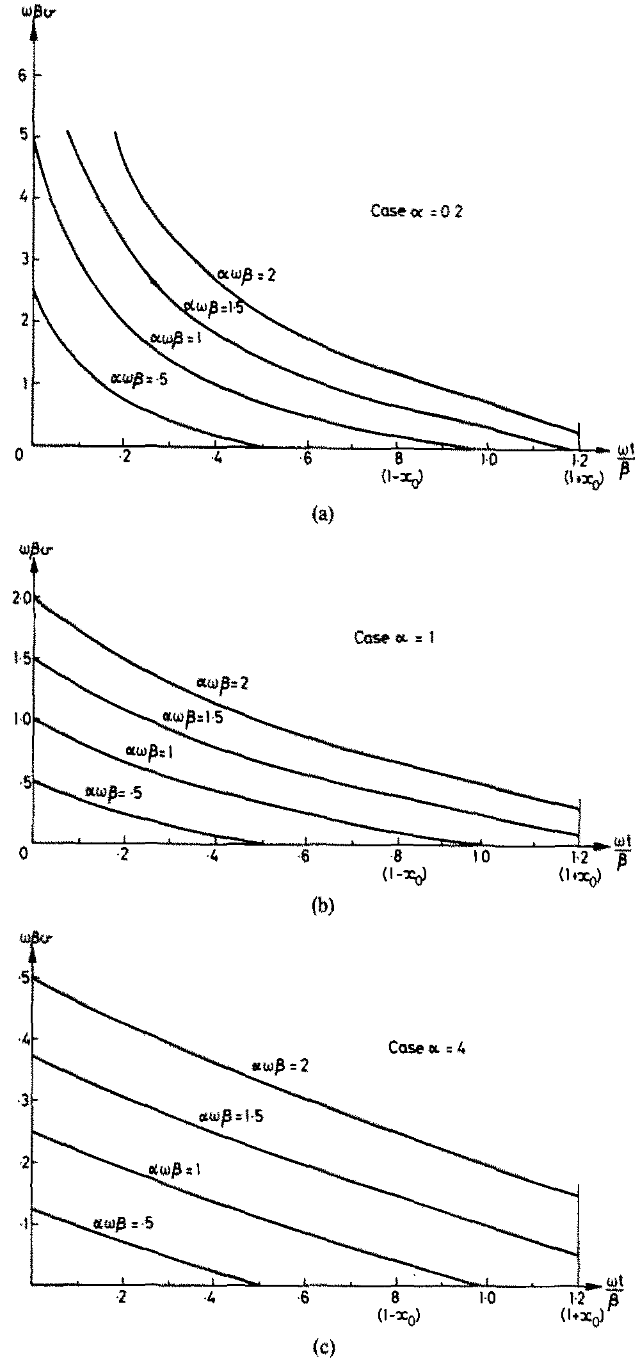

Fig. 6. Variation of the speed v with time t for impact at $x = 0.2$ of a short beam for (a) $\alpha = 0.2$, (b) $\alpha = 1.0$, (c) $\alpha = 4.0$.

condition that v is continuous at $\omega t = \beta(1 + x_0)$, determines v. The functions $g_1(x)$ for $x_0 < x < 1 - 2x_0$, and $k_1(x)$ for $-1 < x < -3x_0$, are then obtained from (5.24) and (5.25). Integral expressions for v, $g_1(x)$ and $k_1(x)$ are readily obtained, but as these are complicated and not very revealing they are not stated explicitly.

It is clear that the complexity of the solution increases rapidly as the deformation progresses, even in the linear strain-hardening case. However, the numerical results given in Fig. 6 and the simpler analysis of Section 4 for the case $x_0 = 0$ shows that multiple reflections occur only when α and $\omega\beta$ are quite large, and that most cases of interest are covered by the comparatively simple analysis of the first two stages of the deformation.

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